King Fahd University of Petroleum & Minerals

Department of Information and Computer Science

Sample Solution

Question	1	2	3	4	5	6	7	8	9	10	11	12	13	14	Total
CLO	3	3	3	3	3	2	2	2	3	3	3	3	3	3	
Max	7	6	6	6	8	10	8	6	7	7	8	8	6	7	**100**
Earned															

Question 1 – Functions [7 Points]

Fill in the first column (**Num**) of the table below by writing the number of the *most* proper text from the 3rd column that is related to the text in the 2nd column.

Num	2nd Column		3rd Column
[2]	$f(n) = n - 1$ from Z to Z	2P	[1] surjective (onto) but not injective
[3]	$f(n) = n^2 + 1$ from Z to Z	1P	[2] bijective: injective and surjective
[4]	$f(n) = n^3$ from Z to Z	1P	[3] not surjective and not injective
[2]	$f(x) = x^3$ from R to R	2P	[4] injective (one-to-one) but not surjective
[1]	$f(n) = \left\lceil \frac{n}{2} \right\rceil$ from Z to Z	1P	

Question 2 – Functions [6 Points]

Let $f(x) = ax + b$ and $g(x) = cx + d$, where a, b, c, and d are constants. Determine necessary and sufficient conditions on the constants a, b, c, and d so that $f \circ g = g \circ f$.

$f \circ g = f(g(x)) = f(cx + d) = a(cx + d) + b = acx + ad + b$ ---- [1] **2P**

$g \circ f = g(f(x)) = g(ax + b) = c(ax + b) + d = acx + bc + d$ ---- [2] **2P**

To determine the necessary and sufficient conditions we let [1] = [2]

$acx + ad + b = acx + bc + d$ which implies $ad + b = bc + d$ **2P**

Which implies $d(a - 1) = b(c - 1)$

Question 3 – Sequences and Summations [6 Points]

Provide a simple formula or rule that generates the terms of an integer sequence that begins with the following list. Assuming that your formula or rule is correct, determine the next three terms of the sequence.

3, 6, 11, 18, 27, 38, 51, 66, 83, 102, ...

The pattern is $a_n = n^2 + 2$. **4P**

The next three terms are 123, 146, 171. **2P**

$a_0 = 3$ **1P**

$a_n = a_{n-1} + 2n + 1$. **3P**

Question 4 – Sequences and Summations [6 Points]

Compute the following double sums.

$$\sum_{m=0}^{2}\sum_{n=0}^{3}(2m+3n)$$

$$= \sum_{m=0}^{2}\left(\sum_{n=0}^{3} 2m + \sum_{n=0}^{3} 3n\right) = \sum_{m=0}^{2}\left(2m\sum_{n=0}^{3} 1 + 3\sum_{n=0}^{3} n\right) = \sum_{m=0}^{2}\left(2m(4) + 3\sum_{n=0}^{3} n\right)$$

$$= \sum_{m=0}^{2}\left(8m + 3\sum_{n=0}^{3} n\right) = \sum_{m=0}^{2}(8m + 3(0+1+2+3)) = \sum_{m=0}^{2}(8m + 3(6))$$ **4P**

$$= \sum_{m=0}^{2}(8m+18)) = \sum_{m=0}^{2}(8m) + \sum_{m=0}^{2}(18) = 8\sum_{m=0}^{2} m + 18\sum_{m=0}^{2} 1$$

$$= 8(0+1+2) + 18(1+1+1) = 8(3) + 18(3) = 24 + 54 = 78$$ **2P**

Question 5 – Cardinality of Sets [8 Points]

Consider the set of the integers that are multiples of 20. Determine whether this set is finite, countably infinite, or uncountable. If it is countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

Countably infinite, since **4P**

4P

n	1	2	3	5	6	7	8	9	10	11	12	...
$f(n)$	0	20	-20	40	-40	60	-60	80	-80	100	-100	...

1P only if negative numbers are not considered

A formula could be used instead – like: $f(n) = (-1)^n \left\lfloor \frac{n}{2} \right\rfloor$ $n \in$ N

Question 6 – Mathematical Induction [10 Points]

Use mathematical induction to prove that

$$2 + 2\,(-7) + 2\,(-7)^2 + \cdots + 2(-7)^n = \frac{1-(-7)^{n+1}}{4}$$

whenever n is a nonnegative integer.

Assume $P(n)$ is $2 + 2\,(-7) + 2\,(-7)^2 + \cdots + 2(-7)^n = \frac{1-(-7)^{n+1}}{4}$ whenever n is a nonnegative integer **1P**

In order to prove this for all integers $n \geq 0$, we first prove the basis step $P(0)$ and then prove the inductive step, that $P(k)$ implies $P(k+1)$.

Basic step: $P(0)$
LHS has just one term which is 2 **2P**
RHS = $\frac{1-(-7)^{0+1}}{4} = \frac{1-(-7)^1}{4} = \frac{1-(-7)}{4} = \frac{8}{4} = 2$
LHS = RHS since 2 = 2 … this proves the basic step.

Inductive step: we assume that $P(k)$ is true that is

$$2 + 2\,(-7) + 2\,(-7)^2 + \cdots + 2(-7)^k = \frac{1-(-7)^{k+1}}{4}$$

2P

We want to show that $P(k+1)$ is true. That is

$$2 + 2\,(-7) + 2\,(-7)^2 + \cdots + 2(-7)^k + 2(-7)^{k+1} = \frac{1-(-7)^{k+2}}{4}$$

$$2 + 2\,(-7) + 2\,(-7)^2 + \cdots + 2(-7)^k + 2(-7)^{k+1}$$

$$= \frac{1-(-7)^{k+1}}{4} + 2(-7)^{k+1} = \frac{1-(-7)^{k+1} + 8(-7)^{k+1}}{4}$$

$$= \frac{1+7(-7)^{k+1}}{4} = \frac{1-(-7)(-7)^{k+1}}{4} = \frac{1-(-7)^{k+2}}{4}$$

4P

This completes the inductive step.
We have proved the basic step and the conductive step. By mathematical induction $P(n)$ is true whenever n is a nonnegative integer.

1P

Question 7 – Recursive Definitions and Structural Induction [8 Points]

Give a recursive definition of the sequence $\{a_n\}$, $n = 1, 2, 3, \ldots$ if

a) $a_n = 4n - 2$

The first 6 terms are 2, 6, 10, 14, 18, 22, ..

$a_1 = 2$ **1P**

$a_n = a_{n-1} + 4$ **3P**

b) $a_n = n^2$

The first 6 terms are 1, 4, 9, 16, 25, 36, ..

$a_1 = 1$ **1P**

$a_n = a_{n-1} + 2(n - 1) + 1$ **3P** or $(\sqrt{a_{n-1}} + 1)^2$

Question 8 – Recursive Definitions and Structural Induction [6 Points]

Give a recursive definition of w^i, where w is a string and i is a nonnegative integer. Where w^i represents the concatenation of i copies of the string w. For example: when w is the string "abc" and i is 4, then w^4 is the string "abcabcabcabc".

Basic Step: $w^0 = \lambda$ **2P**

Recursive Step: $w^n = w(w^{n-1})$ **4P**

Question 9 – Basics of Counting [7 Points]

How many strings are there of Arabic letters of length five or less, not counting the empty string? Assume Arabic has 28 letters.

$28^5 + 28^4 + 28^3 + 28^2 + 28$ **7P**

Question 10 – Basics of Counting [7 Points]

How many positive integers between 100 and 999 inclusive are divisible by 4 but not by 5?

Integers divisible by 4 but not by 5 are integers divisible by 4 that are not integers divisible by 4 and 5. That is divisible by 4 but not divisible by 20 (4 × 5).

So what we need is $n_{\text{by 4 but not 5}} = n_{\text{by 4}} - n_{\text{by 20}}$ **3P**

$$= \frac{(999-100)+1}{4} - \frac{(999-100)+1}{20} = 225 - 45 = 180$$

2P **2P**

Question 11 – The Pigeonhole Principle [8 Points]

Show that if seven integers are selected from the first 10 positive integers, there must be at least two pairs of these integers with the sum 11.

8P

Divide the first ten positive integers into the following 5 groups: {1, 10}, {2, 9}, {3, 8}, {4, 7}, {5, 6}. Notice that the sum of the two numbers in any group is equal to 11.

Proof 1. Since we choose 7 numbers and there are 5 (which is less than 7) groups, by Pigeonhole principle at least two numbers are in the same group. Their sum is 11. Now, besides these 2 numbers in one group, we chose 5 other numbers from 4 other groups. Again, there are more numbers than groups; therefore at least two numbers are in one group. These two numbers form another pair with sum equal to 11.

Proof 2. Since we choose 7 numbers out of 10, only 3 numbers remain unchosen. They can be in a maximum of 3 groups. Therefore at least 2 groups are chosen completely, so we have at least 2 pairs with sum equal to 11.

Question 12 – Permutations and Combinations [8 Points]

A coin is flipped six times where each flip comes up either heads or tails.

a) How many possible outcomes contain exactly three heads?

$$\binom{6}{3} = 20$$

3P

b) How many possible outcomes contain at least three heads?

$$\binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6} = 20 + 15 + 6 + 1 = 42$$

5P

Question 13 – Permutations and Combinations [6 Points]

How many strings of four decimal digits do not contain the same digit twice?

Number of strings of four decimal digits = 10^4

To build a string with 4 distinct digits, we have 10 choices for the first digit, 9 choices for the second digit, 8 choices for the third digit, and 7 choices for the fourth digit. Based on the product rule, we conclude that there are 10 × 9 × 8 × 7 = 5040 four decimal digits that do not contain the same digit twice.

6P

Question 14 – Binomial Coefficients and Identities [7 Points]

What is the coefficient of x^7 in $(2 - x)^{12}$?

Hint: The following is one complete row of Pascal's triangle:

1 10 45 120 210 252 210 120 45 10 1

From this row (10), we need to reach to raw 12.

1	10	45	120	210	252	210	120	45	10	1		
1	11	55	165	330	462	462	330	165	55	11	1	
1	12	66	220	495	792	924	792	495	220	66	12	1

We have $(a + b)^{12}$ where $a = 2$ and $b = -x$.

We are looking for b^7. This will be in the term a^5b^7 when n is 12.

So the coefficient of a^5b^7 is 792. i.e. 792 a^5b^7

Substitute $a = 2$ and $b = -x$.

$(792)\ 2^5(-x)^7$ So the Coefficient of x^7 is –(792) (32) = -25344

Or equiv. 2P

5P